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The PMC Processes Associated to the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli Shift Are Also Bernoulli

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It is proven that the Prigogine–Misra–Courbage (PMC) processes associated to the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift are Bernoulli shifts. The Bernoulli partitions are constructed explicitly by using the decomposition of the transition kernels of the PMC processes on the fibers of the stable manifold of the transformed point (the generating property of these partitions is proved for any two-symbol Bernoulli shifts).

KEY WORDS: Markov processes; Bernoulli shifts; stable manifold; strong convergence to equilibrium; Nelson measure.

1. INTRODUCTION

Let $(X, \mathscr{B}, \mu, T)$ be a Kolmogorov shift. The Prigogine–Misra–Courbage (PMC) processes associated to this dynamical system are Markov processes $V(T)$ depending on T, which preserve μ, and satisfy the following relevant properties[1–3]:

(a) They converge strongly to equilibrium, that is,

$$V(T)^{*n} f \xrightarrow[n \to \infty]{L^2(\mu)} \int f \, d\mu$$

for any $f \in L^2(\mu)$.

(b) Their evolutions commute with the evolution of the dynamical system on the space of densities by means of a nonunitary operator Λ.

(c) There is no loss of information, in the sense that Λ is invertible in a dense subset of densities.

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In this work we study the ergodic theoretical characterization of the PMC processes $V(T)$, that is, the properties of the induced dynamical systems $(X^Z, \mathscr{B}^Z, \mu_{V(T)}, \underline{T})$ where $\underline{T}$ is the shift action on the space of double sequences X^Z and $\mu_{V(T)}$ is the Nelson probability measure[12,13] induced by $V(T)$ and μ on $(X^Z, \mathscr{B}^Z)$. Our main result is the following: when $(X, \mathscr{B}, \mu, T)$ is the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift, the associated PMC processes $(X^Z, \mathscr{B}^Z, \mu_{V(T)}, \underline{T})$ are Bernoulli. We also construct explicit $\mu_{V(T)}$-Bernoulli partitions.

In the proof of this result we use the decomposition of the transition kernel $Q_{V(T)}$ on the fibers $(X_k^+(Tx): k \in Z \cup \{-\infty\})$ of the stable manifold $X^+(Tx)$ of Tx.[4,5] We introduce all these previous results in Section 2. They allow us to construct μ_V-Bernoulli partitions. The generating property of these partitions is proved in Lemma 1 of Section 3 for any $(p, 1-p)$-Bernoulli shifts $(0<p<1)$ and the independent property is shown in Lemma 2.

We remark that the method introduced in Refs. 13 and 14, which allows us to prove that some Markov processes preserving μ are Bernoulli, does not apply to our case. In fact, it is easy to show that in the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift case we have

$$\sup_{|f| \leqslant 1} \left[V(T)^k f - \int f \, d\mu \right] = 1$$

for any $k \geqslant 1$ [where the sup is taken over the $L^1(\mu)$ lattice], while the condition supposed in Ref. 13 is the convergence of the previous quantity to 0 when k increases to infinity.

Our result can be analyzed by means of formula (7) of Section 2. A PMC process consists in choosing [with a probability distribution that for the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift does not depend on the point] some integer m such that the future code of the deterministic evolution is preserved for $n>m$, the past code is selected at random for $n<m$, and for $n=m$ we take a different state. In the $\mu_{V(T)}$-Bernoulli partition given by (9) we stack at time 0 the coordinate m_0 at which the deterministic evolution changes and the past of the orbit $(x(n): n \leqslant m_0)$, which will be lost in the evolution of the PMC process $V(T)$.

2. DESCRIPTION OF THE PMC PROCESSES

2.1. The Canonical PMC Process

Let T be a μ-automorphism of the Lebesgue probability measure space $(X, \mathscr{B}, \mu)$, that is, T is a $\mathscr{B}$-measurable bijection on X that preserves μ. It is a Kolmogorov automorphism if there exists a μ-complete σ-field $\mathcal{a}_0 \subset \mathcal{B}$ such that the sequence $(\mathcal{a}_n = T^n \mathcal{a}_0)_{n \in Z}$ is increasing and exact, the last term meaning that $(\mathcal{a}_n)_{n \in Z}$ is generating: $\mathcal{a}_\infty = \mathcal{B}(\bmod \mu)$, and $\mathcal{a}_{-\infty} = \mathcal{N}(\bmod \mu)$ [where $\mathcal{a}_{\pm\infty} = \lim_{n \to \pm\infty} \mathcal{a}_n$, $\mathcal{N}$ is the trivial σ-field, and a relation $(\bmod \mu)$ means that the relation is verified up to μ-null measure sets]. It is a Bernoulli automorphism if there exists a μ-complete σ-field $\mathcal{F}_0$ such that the sequence of σ-fields $(\mathcal{F}_n = T^n \mathcal{F}_0)_{n \in Z}$ is μ-independent and generating [that is, the σ-field generated by the $(\mathcal{F}_n)_{n \in Z}$ satisfies $\bigvee_{n \in Z} \mathcal{F}_n = \mathcal{B}(\bmod \mu)$]. Any Bernoulli automorphism is Kolmogorov.[7]

A μ-automorphism T induces an operator U_T on $L^2(\mu)$, $U_T f = f \circ T$, which is unitary ($U_T^* = U_T^{-1}$) and μ-Markov: $U_T 1 = 1 = U_T^* 1$, $U_T f \geqslant 0$ if $f \geqslant 0$ (see Ref. 8). Reciprocally, any unitary μ-Markov operator induces a μ-automorphism on $(X, \mathcal{B}, \mu)$.[5] We have $\|U_T^{*n} f\| = \|f\|$ for any $f \in L^2(\mu)$. Then any nontrivial μ-automorphism does not satisfy the condition of strong convergence to equilibrium:

$$U_T^{*n} f \xrightarrow[n \to \infty]{L^2(\mu)} \int f \, d\mu \qquad \text{for} \quad f \in L^2(\mu)$$

Recently in Ref. 1, and in a more general way in Refs. 2 and 3, there has been associated to a Kolmogorov μ-automorphism T a class of operators $V = V(T)$ satisfing

$$V1 = 1 = V^*1, \qquad Vf \geqslant 0 \quad \text{if} \quad f \geqslant 0 \qquad (\mu\text{-Markov property}) \qquad (1a)$$

$$V^{*n} f \xrightarrow[n \to \infty]{} \int f \, d\mu \qquad \text{for any } f \in L^2(\mu)$$

$$\text{(strong convergence to equilibrium)} \qquad (1b)$$

$$\text{there exists a } \mu\text{-Markov operator } \Lambda \text{ such that:} \qquad (1c)$$

$$V^* \Lambda = \Lambda U_T^* \qquad \text{(commutativity relation)} \qquad (1d)$$

$$\Lambda^{-1} \text{ is densely defined in } L^2(\mu)$$

$$\text{(no loss of information)} \qquad (1e)$$

For physical discussions of these conditions see Ref. 1.

It is important to remark that any μ-Markov operator V induces a transition kernel Q_V on $(X, \mathcal{B})$, defined μ-a.e. by the formula $Q_V(x, B) = V 1_B (\bmod \mu)$ and which preserves μ.[10] Reciprocally, any transition probability kernel Q preserving μ induces a μ-Markov operator V_Q on $L^2(\mu)$ by $(V_Q f)(x) = \int f(x) \, Q(x, dx')$ μ-a.e. It is easy to see that any μ-Markov operator V preserves the space of densities

$\mathscr{D}(\mu)=\{f\geqslant 0, \int f\,d\mu=1\}$ and that its adjoint V^*, which is also μ-Markov, is the evolution operator in $\mathscr{D}(\mu)$. This is why conditions (1b) and (1d) are established in terms of the adjoint operators.

Now, for Kolmogorov automorphisms, we shall describe briefly the canonical construction of μ-Markov operators V and Λ that satisfy conditions (1a)–(1e). Let $v=(v_n)_{n\in Z\cup\{\infty\}}$ be a probability vector on $Z\cup\{\infty\}$ satisfying the following assumptions[6]:

$$v_\infty < 1 \tag{2a}$$

$$v_\infty \neq 0 \text{ or there exists a subsequence } n_k \nearrow \infty \text{ such that } v_{n_k}\neq 0 \tag{2b}$$

$$\sum_{n\leqslant 0} n v_n > -\infty \tag{2c}$$

Then it is easy to show that

$$\lambda_n=\prod_{k<n}\left(1-\sum_{m<k} v_m\right)-\prod_{k<n+1}\left(1-\sum_{m<k} v_m\right)$$

is strictly positive for every $n\in Z$ and $\lambda=(\lambda_n)_{n\in Z}$ is a probability vector over Z. Let $\alpha_n=T^n\alpha_0$ (where α_0 satisfies the Kolmogorov property for T). The mean expected value operator E^{α_n} is μ-Markov. Now it can be proved[1 5] that the Markov operators defined in $L^2(\mu)$ by

$$V=\left(\sum_{n\in Z\cup\{\infty\}} v_n E^{\alpha_n}\right) U_T \tag{3a}$$

$$\Lambda=\sum_{n\in Z}\lambda_n E^{\alpha_n} \tag{3b}$$

satisfy conditions (1a)–(1e) (we have written v_n, λ_n instead of $\bar{v}_n$, $\bar{\lambda}_n$ as is done in Refs. 1–3). In this construction Λ is self-adjoint, and it depends on T. For any nonconstant $f\in L^2(\mu)$ the sequence $V^n f$ does not converge to $\int f\,d\mu$ in the $L^2(\mu)$ norm; then we can distinguish V from V^* by means of the conditions of strong convergence to equilibrium. As the operator V gives the time-reversed evolution of V^* (see Ref. 12), the symmetry in time evolution has been broken.

2.2. Description of the Probability Kernel of the PMC Processes

Now we only consider ergodic automorphisms of finite entropy. So, up to isomorphism,[9] we shall only deal with ergodic shifts constructed over some finite alphabet A. Then

$$X=A^Z=\{x=(x(i)\in A\colon i\in Z\},\qquad \mathscr{B}=(\mathscr{P}(A))^Z$$

where $\mathscr{P}(A)$ is the discrete field on A, $(Tx)(i) = x(i+1)$, $\forall i \in Z$, is the shift transformation, and μ is a T-invariant probability measure. Let $X_a = \{x \in X: x(0) = a\}$ and consider the zeroth coordinate partition $\alpha_0 = \{X_a: a \in A\}$ and the past σ-field

$$a_0 = \bigvee_{n \geqslant 0} T^{-n} \hat{\alpha}_0$$

where $\hat{\alpha}_0$ is the field generated by α_0. Then

$$T^n a_0 \underset{n \to \infty}{\nearrow} \mathscr{B}(\text{mod } \mu)$$

and T is a μ-Kolmogorov automorphism iff

$$T^{-n} a_0 \underset{n \to \infty}{\searrow} \mathscr{N}(\text{mod } \mu)$$

In this last case it has been proved[5] that the probability kernel Q_V induced by the Markov operator

$$V = \left(\sum_{n \in Z \cup \{\infty\}} v_n E^{a_n} \right) U_T$$

satisfies

$$Q_V(x, X^+(Tx)) = 1 \tag{4a}$$

$$Q_V(x, \{Tx\}) \geqslant v_\infty \tag{4b}$$

where

$$X^+(y) = \{z \in X: \exists j \in Z \text{ such that } z(i) = y(i), \forall i \geqslant j\} \tag{5}$$

is the stable manifold of $y \in X$. [If we introduce the distance

$$d(x, y) = \sum_{n \in Z} 2^{-|n|} d_s(x(n), y(n))$$

on X, where $d_s(a, a') = 1$ if $a \neq a'$ and 0 if otherwise, we can define the stable manifold of y as

$$X^+(y) = \{z \in X: d(T^n z, T^n y) \xrightarrow[n \to \infty]{} 0\}$$

see Ref. 9.

A more precise result is obtained in the Bernoulli case. Up to isomorphism, we can always assume that the sequence $(T^n \hat{\alpha}_0)_{n \in Z}$ is μ-

independent, so μ is the product measure $\mu=\mu_A^Z$, where $\mu_A=(\mu(X_a)\colon a\in A)$ is a probability vector on A. We suppose $\mu(X_a)>0\ \forall a\in A$. The shift T defined in $(A^Z, (P(A))^Z, \mu_A^Z)$ is called a μ_A-Bernoulli shift.

For $x\in X$ and $-\infty\leqslant i'\leqslant i''\leqslant\infty$ we define the block

$$x(i, i')=(x(i)\colon i'\leqslant i\leqslant i'', i\in Z)$$

and the cylinder

$$[x(i', i'')]=\{y\in X\colon y(i', i'')=x(i', i'')\}$$

The set of finite cylinders $(i', i''\in Z)$ denoted by ξ is a semialgebra generating $\mathscr{B}$.[9] We write $\mu_i(x)=\mu[x(i, i)]$; then for Bernoulli measures we get

$$\mu[x(i', i'')]=\prod_{i=i'}^{i''}\mu_i(x)$$

Now define

$$\tau(x, y)=\inf\{i\in Z\colon x(i+1, \infty)=y(i+1, \infty)\} \tag{6}$$

Then $y\in X^+(x)$ iff $\tau(x, y)<\infty$, $y=x$ iff $\tau(x, y)=-\infty$, and if $\tau(x, y)\in Z$, we have $x(\tau(x, y))\neq y(\tau(x, y))$.

Note $X_k^+(x)=\{y\colon \tau(x, y)=k\}$; then we can partition the stable manifold as follows:

$$X^+(x)=\bigcup_{k\in Z\cup\{-\infty\}} X_k^+(x)$$

[Note that $X_{-\infty}^+(x)=\{x\}$]. Now for Bernoulli measures we have obtained in Ref. 5 the following supplementary properties of the kernel Q_V defined by $V=(\sum_{n\in Z\cup\{\infty\}} v_n E^{a_n})\, U_T$:

$$Q_V(x, X_k^+(Tx))=[\mu_{k+1}(x)^{-1}-1]\sum_{r=k+2}^{\infty} v_{-r}\mu_{k+1}(x)\cdots\mu_{r-1}(x) \tag{7a}$$

$$Q_V(x, \{Tx\})=v_\infty \tag{7b}$$

We can also describe $Q_V(x, C\cap X_k^+(Tx))$ for the cylinders $C\in\xi$. Take $i'\leqslant i''\leqslant k$; note $C=[y(i', i'')]$. In case $i''=k$ we will suppose $y(k)\neq Tx(k)$. We have

$$\begin{aligned} &Q_V(x, [y(i', i'')]\cap X_k^+(Tx))\\ &\quad=\mu([y(i', i'')]\, Q_V(x, X_k^+(Tx))[1-\mu_{k+1}(x)]^{-1} \end{aligned} \tag{7c}$$

This is the characterization of the probability kernel Q_V that will allow us to study the global behaviour of V on the Nelson probability measure space. We must remark that formulas (4a), (4b), and (7a)–(7c) are deduced with no restriction on the coefficients of the probability vector $(v_n)_{n \in Z \cup \{\infty\}}$. Then conditions (2a)–(2c) will not be necessary in the theorem of the following section.

3. THE PMC PROCESSES ARE BERNOULLI

Let V be a μ-Markov operator, where $(X, \mathscr{B}, \mu)$ is a Lebesgue probability space. Let us consider the double sequence space $X^Z = \{\underline{x} = (\underline{x}^t)_{t \in Z}: \underline{x}^t \in X, \forall t \in Z\}$ endowed with the product σ-field $\mathscr{B}^Z$. For a finite sequence of $\mathscr{B}$-sets $[B(t): r \leqslant t \leqslant s]$, we call $\underline{B}(r, s) = \{\underline{x} \in X^Z: \underline{x}^t \in B(t)$ for $r \leqslant t \leqslant s\}$ a cylinder in X^Z. The class of cylinders, which we denote $\underline{\xi}$, is a semialgebra generating $\mathscr{B}^Z$. Define the following (Nelson) probability measure on $(X^Z, \mathscr{B}^Z)$ (see Refs. 12 and 13]):

$$\mu_V(\underline{B}(r, s)) = \int 1_{B(r)} V(1_{B(r+1)} V(\cdots(1_{B(s-1)} V 1_{B(s)})\cdots))\, d\mu \tag{8a}$$

The probability measure space $(X^Z, \mathscr{B}^Z, \mu_V)$ is a Lebesgue one when we complete it. If Q_V is the probability kernel induced by V, we can write (8a) as

$$\mu_V(\underline{B}(r, s)) = \int (1_{B_r}(\underline{x}^r) \int_{B_{r+1}} Q_V(\underline{x}^r, d\underline{x}^{r+1}) \cdots \int_{B_s} Q_V(\underline{x}^{s-1}, d\underline{x}^s))\, d\mu(\underline{x}^r) \tag{8b}$$

Now the shift transformation $\underline{T}$ on X^Z: $(\underline{T}\underline{x})^t = \underline{x}^{t+1}$, preserves μ_V, so $\underline{T}$ is a μ_V-automorphism and it describes the global behavior of the μ-Markov operator V. Now for V defined by (3.1) we obtain [with no restriction on the coefficients of the probability vector $(v_n)_{n \in Z \cup \{\infty\}}$] the following result:

Theorem. Let T be the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift, $\alpha_n = \bigvee_{m \geqslant n} T^{-m} \hat{\alpha}_0$ (where $\hat{\alpha}_0$ is the zeroth coordinate partition of A^Z), $(v_n)_{n \in Z \cup \{\infty\}}$ a probability vector on $Z \cup \{\infty\}$, and $V = (\sum_{n \in Z \cup \{\infty\}} v_n E^{\alpha_n}) U_T$. Then the shift $\underline{T}$ is a Bernoulli μ_V-automorphism on $(A^Z)^Z$.

Proof. If $v_\infty = 1$, we have $V = U_T$ and the result is obvious, so we assume $v_\infty < 1$.

From (4a) (which only needs the Kolmogorov property), we deduce that μ_V is supported by the set $\underline{X} = \{\underline{x} \in X^Z: \underline{x}^{t+1} \in X^+(T\underline{x}^t), \forall t \in Z\}$, that is, $\mu_V(\underline{X})=1$. Let us denote by $\mathscr{B}=\mathscr{B}^Z|_{\underline{X}}$ the restricted σ-field completed by μ_V.

For any $(m, y)\in\Gamma=(Z\cup\{-\infty\})\times A^Z$ we define [recall (6)]

$$Y(m, y)=\{\underline{x}\in\underline{X}\colon \tau(T\underline{x}^0, \underline{x}^1)=m,\ \underline{x}^0(-\infty, m)=y(-\infty, m)\} \tag{9a}$$

[note that $y(-\infty, -\infty)$ is void; then $Y(-\infty, y)=\{\underline{x}\in\underline{X}\colon \underline{x}^1=T\underline{x}^0\}$].

Now the class of sets

$$\mathscr{H}=\{Y(m, y)\colon (m, y)\in\Gamma\} \tag{9b}$$

defines a partition on $\underline{X}$, then a μ_V-partition on X^Z.

Now we shall prove that $\mathscr{H}$ is μ_V-measurable (see Ref. 11). Then the class of sets

$$\hat{\mathscr{H}}=\left\{\bigcup_{(m, y)\in\Gamma'} Y(m, y)\in\mathscr{B}(\text{mod }\mu_V)\colon \Gamma'\subset\Gamma\right\} \tag{10}$$

will be a μ_V-complete σ-field.[11]

We must show that there exists a countable class of sets ζ belonging to $\hat{\mathscr{H}}$ such that for any $Y\neq Y'$ in $\mathscr{H}$ there exists $E\in\zeta$ such that ($Y\subset E$ and $Y'\subset\underline{X}\backslash E$) or ($Y\subset\underline{X}\backslash E$ and $Y'\subset E$). For $m\in Z\cup\{-\infty\}$, $q\in\mathbb{N}\cup\{-1\}$, and $y(m-q, q)\in A^{q+1}$ define

$$E(m, q, y(m-q, q))=\{\underline{x}\in\underline{X}\colon \tau(T\underline{x}^0, \underline{x}^1)=m,\ \underline{x}^0(m-q, m)=y(m-q, m)\}$$

[The choice of $q=-1$ implies $y(m+1, m)$ is void; then the second equality is always verified for any $\underline{x}^0$. If $m=-\infty$, we take $q=-1$.] The class of all these sets $\zeta=\{E(m, q, y(m-q, q))\colon m\in Z\cup\{-\infty\},\ q\in\mathbb{N}\cup\{-1\}, y(m-q, q)\in A^{q+1}\}$ is countable, it is contained in $\hat{\mathscr{H}}$, and it satisfies the above properties for couples $Y\neq Y'$ in $\mathscr{H}$. Then $\mathscr{H}$ is μ_V-measurable.

Our theorem will be deduced from the following two lemmas (in the first one we only need T to be a two-symbol Bernoulli shift).

Lemma 1. Let T be a $(p, 1-p)$-Bernoulli shift $(0<p<1)$. Then $\mathscr{H}$ is generating in $(\underline{X}, \mathscr{B}, \mu_V, \underline{T})$: $\bigvee_{t\in Z}\underline{T}^{-t}\hat{\mathscr{H}}=\mathscr{B}(\text{mod }\mu_V)$.

Proof. Let $s'\leqslant s''$ in Z, and for any $s'\leqslant s\leqslant s''$ take a finite cylinder $[z_s(i'_s, i''_s)]$ in A^Z, where $i'_s\leqslant i''_s$. Now $C=\{\underline{x}\in\underline{X}\colon \underline{x}^s\in[z_s(i'_s, i''_s)], s'\leqslant s\leqslant s''\}$ is a cylinder in $(A^Z)^Z$ and the class of them generates $\mathscr{B}^Z$. Since

$$C=\bigcap_{s=s'}^{s''}\ \bigcap_{i=i'_s}^{i''_s}\{\underline{x}\in\underline{X}\colon \underline{x}^s(i)=z_s(i)\}$$

the assertion of the lemma will be established when we show

$$\{\underline{x} \in \underline{X}: \underline{x}^s(i) = a\} \in \bigvee_{t \in Z} \underline{T}^{-t}\hat{\mathscr{H}} (\text{mod } \mu) \text{ for any } \quad s, i \in Z, \quad a \in A \tag{11}$$

Let us fix s, $i \in Z$ and $a \in A$ and call $C = \{\underline{x} \in \underline{X}: \underline{x}^s(i) = a\}$.

Let $Y(m, y) \in \mathscr{H}$; then

$$\underline{T}^{-t} Y(m, y) = \{\underline{x}: \tau(T\underline{x}^t, \underline{x}^{t+1}) = m, \underline{x}^t(-\infty, m) = y(-\infty, m)\}$$

So, an atom of the partition $\bigvee_{t=s}^{u} T^{-t}\hat{\mathscr{H}}$ is given by

$$\begin{aligned} Y &= Y(m_t, y_t; s \leqslant t \leqslant u) \\ &= \{\underline{x} \in \underline{X}: \tau(T\underline{x}^t, \underline{x}^{t+1}) = m_t, \underline{x}^t(-\infty, m_t) = y_t(-\infty, m_t), s \leqslant t \leqslant u\} \end{aligned}$$

Note that the equality $\tau(T\underline{x}^t, \underline{x}^{t+1}) = m_t$ is equivalent to $\underline{x}^t(i) = \underline{x}^{t+1}(i-1)$ for $i > m_t + 1$ and $\underline{x}^t(m_t + 1) \neq \underline{x}^{t+1}(m_t)$. Since A is a two-symbol set (we can take $A = \{-1, 1\}$), the last relation can be written as $\underline{x}^t(m_t + 1) = -\underline{x}^{t+1}(m_t)$.

Let $L(C) = \{Y \in \bigvee_{t \geqslant s} \underline{T}^{-t}\hat{\mathscr{H}}: Y \subset C\}$. Define $D = \bigcup_{Y \in L(C)} Y$. Since D belongs to $\bigvee_{t \geqslant s} \underline{T}^{-t}\hat{\mathscr{H}}$, it suffices for us to show the equality $C = D (\text{mod } \mu_V)$. Let

$$K(i, s) = \{Y(m_t, y_t; t \geqslant s): m_t < i - t + s \text{ for any } t \geqslant s\}$$

We shall show that $D' = \bigcup_{Y \in K(i,s)} Y$ satisfies $(C \backslash D) \subset D'$ and $\mu_V(D') = 0$. This will finish the proof, since $D \subset C$.

The inclusion $(C \backslash D) \subset D'$ is equivalent to $(\underline{X} \backslash C) \cup D \supset (\underline{X} \backslash D')$; so, we must prove that any $Y = Y(m_t, y_t; t \geqslant s) \notin K(i, s)$ satisfies $Y \subset \underline{X} \backslash C$ or $Y \subset C$. Let us suppose $Y \cap C \neq \phi$; we shall prove $Y \subset C$.

Take $t_0 = \inf\{t \geqslant s: m_t \geqslant i - (t - s)\}$; then $m_t < i - t + s$ for $s \leqslant t < t_0$ and $m_{t_0} \geqslant i - t_0 + s$. The coordinate

$$\underline{x}^{t_0}(i - t_0 + s) = y_{t_0}(i - t_0 + s)$$

is fixed. If $i - t_0 + s + 1 > m_{t_0 - 1} + 1$, we have

$$\underline{x}^{t_0 - 1}(i - t_0 + s + 1) = \underline{x}^{t_0}(i - t_0 + s) = y_{t_0}(i - t_0 + s)$$

In the other possible case $i - t_0 + s + 1 = m_{t_0 - 1} + 1$ we have

$$\underline{x}^{t_0 - 1}(i - t_0 + s + 1) \neq \underline{x}^{t_0}(i - t_0 + s)$$

Then

$$\underline{x}^{t_0 - 1}(i - t_0 + s + 1) = -y_{t_0}(i - t_0 + s)$$

Let us define

$$h(t_0, i-t_0+s) = \operatorname{card}\{s \leqslant t \leqslant t_0 : i-t+s = m_t + 1\}$$

which counts the number of times we change the sign of the symbol $y_{t_0}(i-t_0+s)$ when we shift it from coordinate t_0 to coordinate s. A simple induction gives us

$$\underline{x}^s(i) = (-1)^{h(t_0, i-t_0+s)}\, y_{t_0}(i-t_0+s)$$

that is, $\underline{x}^s(i)$ is fixed in Y. Since there exists some point $\tilde{\underline{x}} \in Y \cap C$, we have $\tilde{\underline{x}}^s(i) = a$, and then any $\underline{x} \in Y$ satisfies $\underline{x}^s(i) = a$, so $Y \subset C$.

Now let us prove $\mu_V(D') = 0$. Let

$$K(i, s, u) = \{Y(m_t, y_t; s \leqslant t \leqslant u) : m_t < i-t+s \text{ for } s \leqslant t \leqslant u\}$$

The sequence

$$D'(u) = \bigcup_{Y \in K(i, s, u)} Y$$

decreases to D' as $u \to \infty$. So it suffices to prove $\mu_V(D'(u)) \to_{u \to \infty} 0$. We have

$$D'(u) = \{\underline{x} : \tau(T\underline{x}^t, \underline{x}^{t+1}) < i-t+s, \forall s \leqslant t \leqslant u\}$$

so

$$\mu_V(D'(u)) = \int \Bigg[\int_{M_i(\underline{x}^s)} Q_V(\underline{x}^s, d\underline{x}^{s+1})$$
$$\times \int_{M_{i-1}(\underline{x}^{s+1})} \cdots \int_{M_{i-u+s}(\underline{x}^u)} Q_V(\underline{x}^u, d\underline{x}^{u+1})\Bigg]\, d\mu(\underline{x}^s)$$

where

$$M_j(x) = \Bigg[\bigcup_{-\infty < k < j} X_k^+(Tx)\Bigg] \cup \{Tx\}$$

[see after (6)].

Let $R_j = \sup_{z \in X} Q_V(z, M_j(z))$; then $\mu_V(D'(u)) \leqslant \prod_{j=i-u+s}^{i} R_j$. Since R_j decreases when j decreases, it suffices to show the inequality $\lim_{j \to -\infty} R_j < 1$ to obtain the result. Now

$$Q_V(z, M_j(z)) = \sum_{-\infty < k < j} Q_V(x, X_k^+(T_x)) + v_\infty$$

Let $\underline{c}=\inf\{\mu[c]: c\in A\}$, $\bar{c}=\sup\{\mu[c]: c\in A\}$ (we have $0<\underline{c}\leqslant\bar{c}<1$). By using formula (7a) we get

$$(R_j-v_\infty)(1-\underline{c})^{-1}\leqslant \sum_{k\leqslant j-1}\ \sum_{r\geqslant k+2} v_{-r}\bar{c}^{r-(k+2)}$$

and if we develop this last term we arrive to the inequality

$$(R_j-v_\infty)(1-\underline{c})^{-1}\leqslant \sum_{-\infty<r\leqslant j+1} v_{-r}+\sum_{j+1\leqslant r<\infty} v_{-r}\bar{c}^{r-(n+1)}$$

The right-hand side converges to 0 when j decreases to $-\infty$; hence, $\lim_{j\to-\infty} R_j=v_\infty<1$. ∎

Now we shall prove the independence property when $p=1-p=\frac{1}{2}$:

Lemma 2. Let T be a $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift. Then the sequence of σ-fields $(\underline{T}^{-t}\hat{\mathscr{H}}: t\in Z)$ is μ_V-independent.

Proof. We must show that for any $s<u$ in Z the sequence $(\underline{T}^{-t}\hat{\mathscr{H}}: s\leqslant t\leqslant u)$ is μ_V-independent. Let

$$\begin{aligned}E_t&=E_t(m_t, q_t, y_t(m_t-q_t, m_t))\\&=\{\underline{x}: \tau(T\underline{x}^t, \underline{x}^{t+1})=m_t, \underline{x}^t(m_t-q_t, m_t)=y_t(m_t-q_t, m_t)\}\end{aligned}$$

for $m_t\in Z\cup\{-\infty\}, q_t\geqslant -1$, $y_t(m_t-q_t, m_t)\in A^{q_t+1}$ [we take $q_t=-1$ when $m_t=-\infty$; recall that $q_t=-1$ means $y_t(m_t+1, m_t)$ is void; thus, the last equality in the definition of E_t is always satisfied]. Let $E=\bigcap_{t=s}^u E_t$; then Lemma 2 will follow when we show $\mu_V(E)=\prod_{t=s}^u \mu_V(E_t)$ [for any choice of $s<u$ and $(E_t: s\leqslant t\leqslant u)$].

Take the block $C_t=[y_t(m_t-q_t, m_t)]$ (it is eventually the set X if $q_t=-1$). We have

$$\mu_V(E_t)=\int 1_{C_t}(\underline{x}^t)\, Q_V(\underline{x}^t, X^+_{m_t}(T\underline{x}^t))\, d\mu(\underline{x}^t)$$

Since $\mu_r(z)=\frac{1}{2}$ for any $z\in X$ [because T is the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift], we get $Q_V(z, X_k^+(Tz))=S_k$, where

$$S_k=\sum_{k+2\leqslant r<\infty} v_{-r}2^{(k+1)-r}$$

for $k\in Z$ and $S_{-\infty}=v_\infty$ [we have used equality (7a)]. Then

$$\mu_V(E_t)=S_{m_t}\mu(C_t)=S_{m_t}\prod_{i=m_t-q_t}^{m_t}\mu_i(y)=S_{m_t}2^{-(q_t+1)}$$

Now we shall write E in a suitable form in order to evaluate $\mu_V(E)$.

For any $s \leqslant t \leqslant u$ such that $q_t \neq -1$ and any $m_t - q_t \leqslant i \leqslant m_t$ define

$$\Theta(t, i) = \inf\{s \leqslant t' \leqslant t\text{: for any } t' \leqslant t'' \leqslant t \text{ we have } i + t - t'' > m_{t''}\}$$

[so we can shift the restriction $y_t(i)$ from the coordinate t to the coordinate $\Theta(t, i)$]. For $m_s - q_s \leqslant i \leqslant m_s$ take $\Theta(s, i) = s$. The function

$$(t, i) \to \Psi(t, i) = (\Theta(t, i), i + t - \Theta(t, i))$$

is one-to-one, so if we denote $\eta(t) = \text{card}[\Psi^{-1}(\{t\} \times Z)]$ (the number of restrictions such that their shifting to the left finishes in t), we get

$$\sum_{t=s}^{u} \eta(t) = \sum_{t=s}^{u} (q_t + 1)$$

For any $s < t \leqslant u$ we have $i + t - \Theta(t, i) > m_{\Theta(t,i)}$ and, if $\Theta(t, i) > s$, we have

$$i + t - \Theta(t, i) \leqslant m_{\Theta(t,i)-1} - 1$$

Let

$$h(t, i) = \text{card}\{\Theta(t, i) \leqslant t'' \leqslant t\text{: } i + t - t'' = m_{t''} + 1\}$$

be the number of times the restriction $y_t(i)$ changes sign when shifted from t to $\Theta(t, i)$. Let $s \leqslant t' \leqslant u$. Now define the cylinder

$$F(t') = \{z \in X\text{: } z(i + t - \Theta(t, i)) = (-1)^{h(t,i)}\, y(t, i)$$
$$\text{for any } (t, i) \in \Psi^{-1}(\{t'\} \times Z)\}$$

Define $F(u+1) = X$, $\eta(u+1) = 0$. For $s \leqslant t \leqslant u$ take

$$G_{t+1}(\underline{x}^t) = X^{+}_{m_t}(T\underline{x}^t) \cap F(t+1)$$

From formula (7c) we get

$$Q_V(\underline{x}^t, G_{t+1}(\underline{x}^t)) = S_{m_t} 2^{-\eta(t+1)}$$

Since

$$E = \{\underline{x}\text{: } \underline{x}^s \in F(s),\ \underline{x}^{t+1} \in G_{t+1}(\underline{x}^t) \text{ for any } s \leqslant t \leqslant u\}$$

we get

$$\begin{aligned}\mu_V(E) &= \int \left[1_{F(s)}(\underline{x}^s) \int_{G_{s+1}(\underline{x}^s)} Q(\underline{x}^s, d\underline{x}^{s+1}) \right. \\ &\qquad \left. \times \cdots \int_{G_{u+1}(\underline{x}^u)} Q(\underline{x}^u, d\underline{x}^{u+1}) \right] d\mu(\underline{x}^s) \\ &= \left(\prod_{t=s}^{u} S_{m_t} \right) 2^{-\sum_{t=s}^{u} \eta(t)} = \prod_{t=s}^{u} \mu_V(E_t).\end{aligned}$$

Then Lemma 2 is verified and the theorem follows. ∎

Since $\underline{T}$ is a Bernoulli μ_V-automorphism, it is completely characterized by its entropy. We shall prove it is infinite for any Bernoulli shift T when $v_\infty < 1$:

Proposition. Let T be a Bernoulli shift, and

$$V = \left(\sum_{n \in Z \cup \{\infty\}} v_n E^{a_n} \right) U_T$$

where $(v_n)_{n \in Z \cup \{\infty\}}$ is a probability vector such that $v_\infty < 1$. Then the Kolmogorov–Sinai entropy of $(X^Z, \mathscr{B}^Z, \mu_V, \underline{T})$ is infinite: $h_{\mu_V}(\underline{T}) = \infty$.

Proof. It is easy to see that for any Markov operator V we have

$$h_{\mu_V}(\underline{T}) \geqslant \int H_{Q_V(x,\cdot)}(\mathscr{B})\, d\mu(x)$$

where $H_{Q_V(x,\cdot)}(\mathscr{B})$ is the entropy of the space $(X, \mathscr{B})$ with respect to the measure $Q_V(x, \cdot)$ defined on it. From (7c) it results that $Q_V(x, \cdot)$ is non-atomic, so $H_{Q_V(x,\cdot)}(\mathscr{B}) = \infty$ for any x. Hence $h_{\mu_V}(\underline{T}) = \infty$. ∎

Remark 1. Let $(v^{(m)})_{m \geqslant 1}$ be a sequence of probability measures on $Z \cup \{\infty\}$ such that:

(a) Each one of them satisfies conditions (2a)–(2c).

(b) They converge weakly to a measure supported by ∞, that is, $v_n^{(m)} \to_{m \to \infty} 0$ for any $n \in Z$, $v_\infty^{(m)} \to_{m \to \infty} 1$.

Let

$$V^{(m)} = \left(\sum_{n \in Z \cup \{\infty\}} v_n^{(m)} E^{a_n} \right) U_T$$

be our canonical PMC processes [with T the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift]. Then $(\mu_{V^{(m)}})_{m \geqslant 1}$ is a sequence of Bernoulli measures of infinite entropy on $(X^Z, \mathscr{B}^Z)$ converging weakly to the Bernoulli measure μ_{U_T}, which has finite entropy equal to log 2.

Remark 2. It is direct to show that the systems $(X^Z, \mathscr{B}^Z, \mu_V, \underline{T}^{-1})$ and $(X^Z, \mathscr{B}^Z, \mu_{V^*}, \underline{T})$ are isomorphic [in fact, the time-reversing function $\Phi((\underline{x}^t)_{t \in Z}) = (\underline{x}^{-t})_{t \in Z}$ is the isomorphism; see Ref. 12]. Let T be the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift. Since $(X^Z, \mathscr{B}^Z, \mu_V, \underline{T})$ is Bernoulli, it is isomorphic to its inverse, hence isomorphic to $(X^Z, \mathscr{B}^Z, \mu_{V^*}, \underline{T})$; while the dynamics of the Markov operators V and V^* can be distinguished by the property of strong convergence to equilibrium.

Remark 3. After this manuscript was submitted, I received a private communication from Prof. Sheldon Goldstein, who extends the main result of this paper:

Theorem. Let $(X, \mathscr{B}, \mu, T)$ be a Bernoulli shift and

$$V=\left(\sum_{n\in Z\cup\{\infty\}} v_n E^{\alpha_n}\right) U_T$$

Then the shift $(X^Z, \mathscr{B}^Z, \mu_V, \underline{T})$ is Bernoulli.

Proof. Let $\hat{X}=(Z\cup\{-\infty\})\times X$ endowed with the product σ-field $\hat{\mathscr{B}}=\mathscr{P}(Z\cup\{-\infty\})\otimes\mathscr{B}$. Define the following transition kernel:

$$\hat{Q}((k, x), \{n\}\times B)=v_{-n}((E^{\alpha-k}U_T)\,1_A)(x)$$

It is easy to show that Q preserves $\hat{\mu}=\hat{v}\otimes\mu$, where $\hat{v}(n)=v_{-n}$. Then $\hat{Q}$ induces a $\hat{\mu}$-Markov operator $\hat{V}$, $\hat{V}1_{\{n\}\times B}(k, x)=\hat{Q}((k, x), \{n\}\times B))$. Now consider the shift $\hat{T}$ on $(\hat{X}^Z, \hat{\mathscr{B}}^Z, \hat{\mu}_{\hat{V}})$. It is not hard to prove that the σ-field generated by the atoms $\{\{k\}\times x^0(-\infty, k-1)\}$ is $\hat{\mu}_{\hat{V}}$-independent and generating, so the shift $\hat{T}$ is $\hat{\mu}_{\hat{V}}$-Bernoulli.

Now the projection

$$\begin{array}{rccc} \phi: & \hat{X}^Z & \to & X^Z \\ & (k^t, x^t)_{t\in Z} & \to & (x^t)_{t\in Z} \end{array}$$

satisfies $\phi\hat{\mu}_{\hat{V}}=\mu_V$ and $\phi\circ\hat{T}=\underline{T}\circ\phi$. Then $(X^Z, \mathscr{B}^Z, \mu_V, \underline{T})$ is a factor of $(\hat{X}^Z, \hat{\mathscr{B}}^Z, \hat{\mu}_{\hat{V}}, \hat{T})$. Since factors of Bernoulli shifts are also Bernoulli (see Ornstein[15]), we deduce the result. ∎

The above proof, while short and elegant, does not provide an explicit μ_V-Bernoulli partition for a general Bernoulli measure μ (as in the $(\frac{1}{2}, \frac{1}{2})$ case), which remains an open problem.

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